

Scaling asymptotics for quantized Hamiltonian flows

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Abstract

In recent years, the near diagonal asymptotics of the equivariant components of the Szegő kernel of a positive line bundle on a compact symplectic manifold have been studied extensively by many authors. As a natural generalization of this theme, here we consider the local scaling asymptotics of the quantization of a Hamiltonian symplectomorphism, and specifically how they concentrate on the graph of the underlying classical map.

1 Introduction

Suppose that M is a connected d -dimensional complex projective manifold, and let (B, h) be a positive Hermitian line bundle on M . Thus B is ample as an holomorphic line bundle, and h is an Hermitian metric on B , such that the unique compatible connection ∇ on (B, h) has curvature $-2i\omega$, where ω is a Kähler form. If B^\vee is the dual line bundle, let $B^\vee \supseteq X \xrightarrow{\pi} M$ be the unit circle bundle; the connection 1-form α is a contact form on X .

These choices determine natural volume forms $dV_M =: (1/d!)\omega^{\wedge d}$ on M and $d\mu_X =: (1/2\pi)\alpha \wedge \pi^*(dV_M)$ on X , respectively, hence induce Hermitian structures on the vector spaces $H^0(M, B^{\otimes k})$ of global holomorphic sections of the tensor powers $B^{\otimes k}$, for $k = 0, 1, 2, \dots$. The Hardy space $H(X) \subseteq L^2(X)$ is unitarily isomorphic in a natural manner to the Hilbert space direct sum of the $H^0(M, B^{\otimes k})$'s, and $H^0(M, B^{\otimes k})$ corresponds to the k -th isotype $H(X)_k \subseteq H(X)$ for the circle action on X [SZ].

In geometric quantization, the symplectic manifold $(M, 2\omega)$ is viewed as a ‘classical phase space’, and the Hilbert spaces $H^0(M, B^{\otimes k})$ as corresponding ‘quantum spaces’ at Planck’s constant $\hbar = 1/k$; the semiclassical regime

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corresponds to letting $k \rightarrow +\infty$. A basic theme in this setting is the quantization of Hamiltonian functions and their Hamiltonian flows (see, e.g., [B], [BG], [Z1]).

Consider a classical observable on M , given by a \mathcal{C}^∞ function $f : M \rightarrow \mathbb{R}$, with Hamiltonian vector field v_f , and corresponding flow $\phi_\tau^M : M \rightarrow M$ ($\tau \in \mathbb{R}$); thus $\phi^M : \tau \mapsto \phi_\tau^M$ is a 1-parameter group of Hamiltonian symplectomorphisms. One regards the self-adjoint Toeplitz operator associated to f , $T_f : H(X) \rightarrow H(X)$, as the quantization of f ; explicitly, $T_f =: \Pi \circ M_f \circ \Pi$, where $\Pi : L^2(X) \rightarrow H(X)$ is the orthogonal projection (the so-called Szegő projector), and M_f is multiplication by f (pulled-back to X). Being S^1 -invariant, T_f restricts to ‘quantum observables’ $T_f^{(k)} : H(X)_k \rightarrow H(X)_k$. On the other hand, the quantization of ϕ_τ^M should be a family of S^1 -invariant unitary operators $\Phi_\tau : H(X) \rightarrow H(X)$, asymptotically related to the dynamics of ϕ_τ^M .

There exists a contact vector field \tilde{v}_f on X lifting v_f [?], which depends on f (and not just on v_f). Consequently, ϕ^M lifts to a 1-parameter group $\phi^X : \tau \mapsto \phi_\tau^X$ of contactomorphisms of (X, α) ; pull-back determines a unitary action $(\phi_{-\tau}^X)^* : L^2(X) \rightarrow L^2(X)$.

When ϕ_τ^M is holomorphic, $(\phi_{-\tau}^X)^*$ preserves $H(X)$, and the restriction is a quantization of ϕ^M . Thus one sets $\Phi_\tau =: \Pi \circ (\phi_{-\tau}^X)^* \circ \Pi : H(X) \rightarrow H(X)$ in this case.

However, in general $(\phi_{-\tau}^X)^*(H(X)) \subsetneq H(X)$, and $\Pi \circ (\phi_{-\tau}^X)^* \circ \Pi : H(X) \rightarrow H(X)$ is not unitary. Nonetheless, in the setting of Fourier-Hermite distributions [BG], Zelditch proved that there exists a canonical family of invariant zeroth order Toeplitz operators $R_\tau : H(X) \rightarrow H(X)$, such that $\Phi_\tau =: R_\tau \circ \Pi \circ (\phi_{-\tau}^X)^* \circ \Pi$ is indeed a unitary automorphisms of $H(X)$ (essentially), and computed the leading symbol of R_τ by the symbolic calculus of symplectic spinors [Z1].

Here we approach similar issues by the local scaling asymptotics of the distributional kernels of operators of the same general form as Φ_τ . Much attention has been drawn in recent years by the near-diagonal scaling asymptotics of the equivariant components of Szegő kernels, involving various authors and points of view; this paper is specifically related to the approach in [BSZ], [SZ], based on the microlocal theory of [BS] (see for instance [MM1] and [MM2] for a different perspective). This line of research originated from the so-called TYZ expansion, which first appeared in [T], [C], [Z2].

We shall first build on [BSZ] and [SZ] to determine the equivariant scaling asymptotics of

$$U_\tau =: R_\tau \circ \Pi \circ (\phi_{-\tau}^X)^* \circ \Pi, \quad (1)$$

now over the graph of ϕ_τ^M (Theorems 1.1 and 1.2); here R_τ is a general \mathcal{C}^∞

family of invariant zeroth order Toeplitz operators on X . Then we shall determine the leading symbol of R_τ if U_τ is unitary (Corollary 1.1); in the reverse direction, we shall derive a version of the Zelditch unitarization Lemma in [Z1] (Corollary 1.2). To leading order, U_τ solves a Schrödinger type equation (Proposition 1.1).

By definition, $R_\tau = \Pi \circ Q_\tau \circ \Pi$, where Q_τ is a zeroth order invariant pseudodifferential operator of classical type on X ; the symbol ϱ_τ of R_τ is the restriction of the symbol of Q_τ to the closed symplectic cone sprayed by the connection form,

$$\Sigma =: \{(x, r\alpha_x) : x \in X, r > 0\} \subseteq T^*X \setminus \{0\}.$$

Being homogeneous of degree zero and S^1 -invariant, ϱ_τ is really a smooth function on M .

Define U_τ by (1). Identifying densities, half-densities and functions by the previous choices, also denote by $U_\tau \in \mathcal{D}'(X \times X)$ the Schwartz kernel of U_τ . By invariance, U_τ restricts to operators $U_{\tau,k} : H(X)_k \rightarrow H(X)_k$. If $k = 0, 1, 2, \dots$ and $\{s_{kj}\}_{j=1}^{N_k}$ is an orthonormal basis of $H(X)_k$, the corresponding distributional kernels are $U_{\tau,k} = \sum_{j=1}^{N_k} U_\tau(s_{kj}) \boxtimes \overline{s_{kj}} \in \mathcal{C}^\infty(X \times X)$, and $U_\tau(x, y) = \sum_{k \geq 0} U_{\tau,k}(x, y)$. More explicitly,

$$U_{\tau,k}(x, y) = \sum_{j=1}^{N_k} U_\tau(s_{kj})(x) \cdot \overline{s_{kj}(y)}.$$

As in the case of the Szegö kernel, the following scaling asymptotics for $U_{\tau,k}$ are expressed in terms of Heisenberg local coordinates on X ; these are precisely defined in [SZ]. A system of Heisenberg local coordinates centered at some $x \in X$ is built of a system of preferred local coordinates on M , centered at $m = \pi(x)$ (meaning that the symplectic and complex structure are the standard ones at the origin), and a preferred local section e_L of L at $m = \pi(x)$ (this is a prescription on the second order jet of e_L at m). In particular, a system of Heisenberg local coordinates centered at x induces unitary isomorphism $T_m M \cong \mathbb{C}^d$ and $T_x X \cong \mathbb{R} \times \mathbb{C}^d$; in the latter, $\mathbb{R} \times \{\mathbf{0}\}$ and $\{0\} \times \mathbb{C}^d$ correspond to the vertical and horizontal tangent spaces, respectively.

In Heisenberg local coordinates, the equivariant scaling asymptotics of Szegö kernels exhibit their universal nature. If γ is a system of Heisenberg local coordinates centered at x , following [SZ] we shall let $x + (\theta, \mathbf{v}) =: \gamma(\theta, \mathbf{v})$ if $\theta \in (-\pi, \pi)$, and $\mathbf{v} \in \mathbb{R}^{2d}$ is sufficiently small; in the same range, we shall also write $x + \mathbf{v} =: x + (0, \mathbf{v})$. If $\vartheta \in (-\pi, \pi)$ the action $r_\vartheta : X \rightarrow X$ of $e^{i\vartheta} \in S^1$ is expressed by a translation by ϑ in the angular coordinate, that

is, $r_\vartheta(x + (\theta, \mathbf{v})) = x + (\vartheta + \theta, \mathbf{v})$ wherever defined; furthermore, $m + \mathbf{v} =: \pi(x + (\theta, \mathbf{v}))$ is the underlying system of preferred local coordinates at m . Given the built-in unitary isomorphism $T_m M \cong \mathbb{C}^d$, we shall also use the expressions $x + (\theta, \mathbf{v})$ and $x + \mathbf{v}$ for $\mathbf{v} \in T_m M$ of suitably small norm.

Fix $x \in X$ and $\tau \in \mathbb{R}$, and set $x_\tau =: \phi_{-\tau}^X(x)$; if $m =: \pi(x)$ and $m_\tau =: \pi(x_\tau)$, then $m_\tau = \phi_{-\tau}^M(m)$. Choose Heisenberg local coordinates centered at x and x_τ respectively; then the differential $d_m \phi_{-\tau}^M : T_m M \rightarrow T_{m_\tau} M$ corresponds to a $2d \times 2d$ symplectic matrix $A_{\tau, m}$. A change in Heisenberg local coordinates at x and x_τ will turn $A_{\tau, m}$ into $A'_{\tau, m} =: R A_{\tau, m} S^t$, where R and S are unitary (that is, symplectic and orthogonal).

Definition 1.1. For $\tau \in \mathbb{R}$, the *saturated graph* of $\phi_{-\tau}^X$ is

$$\widetilde{\text{graph}}(\phi_{-\tau}^X) =: (\pi \times \pi)^{-1}(\text{graph}(\phi_{-\tau}^M)) \subseteq X \times X.$$

Thus $\widetilde{\text{graph}}(\phi_{-\tau}^X)$ is the saturation of $\text{graph}(\phi_{-\tau}^X)$ under the S^1 -action. In other words, $(x, y) \in \widetilde{\text{graph}}(\phi_{-\tau}^X)$ if and only if $y = r_{\vartheta}(x_\tau)$ for some $e^{i\vartheta} \in S^1$.

As $k \rightarrow +\infty$, the kernel $U_{\tau, k}$ concentrates on $\widetilde{\text{graph}}(\phi_{-\tau}^X)$, meaning that $U_{\tau, k}(x, y) = O(k^{-\infty})$ uniformly in $(x, y) \in X \times X \setminus \widetilde{\text{graph}}(\phi_{-\tau}^X)$. More precisely, if dist_X is the Riemannian distance on X then we have:

Theorem 1.1. For any $D, \varepsilon > 0$, uniformly in $(x, y) \in X \times X$ satisfying $\text{dist}_X(y, S^1 \cdot x_\tau) \geq D k^{\varepsilon - \frac{1}{2}}$, we have $U_{\tau, k}(x, y) = O(k^{-\infty})$ as $k \rightarrow +\infty$.

Let us analyze the rate at which $U_{\tau, k}$ concentrates on the saturated graph. For any $e^{i\vartheta_1}, e^{i\vartheta_2} \in S^1$ and $(x, y) \in X \times X$, we have $U_{\tau, k}(r_{\vartheta_1}(x), r_{\vartheta_2}(y)) = e^{ik(\vartheta_1 - \vartheta_2)} U_{\tau, k}(x, y)$, so that without loss we may work in the neighborhood of a given $(x, x_\tau) \in \text{graph}(\phi_{-\tau}^X)$. Thus we may consider the behavior of $U_{\tau, k}$ at points of the form $(x + (\vartheta_1, \mathbf{u}), x_\tau + (\vartheta_2, \mathbf{w}))$, computed in systems of Heisenberg local coordinates centered at x and x_τ , respectively. Now

$$\begin{aligned} & U_{\tau, k}(x + (\vartheta_1, \mathbf{u}), x_\tau + (\vartheta_2, \mathbf{w})) \\ &= U_{\tau, k}(r_{\vartheta_1}(x + \mathbf{u}), r_{\vartheta_2}(x_\tau + \mathbf{w})) = e^{ik(\vartheta_1 - \vartheta_2)} U_{\tau, k}(x + \mathbf{u}, x_\tau + \mathbf{w}), \end{aligned} \tag{2}$$

so we need only consider pairs $(x + \mathbf{u}, x_\tau + \mathbf{w})$ converging to (x, x_τ) .

In order to formulate our result, we need to define a certain quadratic function $\mathcal{S}_A : \mathbb{R}^{2d} \times \mathbb{R}^{2d} \rightarrow \mathbb{C}$ associated to a symplectic matrix A . Let

$$J_0 =: \begin{pmatrix} 0 & -I_d \\ I_d & 0 \end{pmatrix};$$

thus J_0 represents the standard complex structure on \mathbb{R}^{2d} , and $-J_0$ the standard symplectic structure ω_0 .

Definition 1.2. Let A be a symplectic $2d \times 2d$ matrix, and let $A = O P$ be its polar decomposition; thus O is orthogonal and symplectic, hence unitary, and P is symmetric positive definite and symplectic. Then the following matrices are symmetric:

$$Q_A = I + P^2, \quad \mathcal{P}_A =: O Q_A^{-1} O^t, \quad \mathcal{R}_A =: O (I - P^2) Q_A^{-1} J_0 O^t.$$

For $\mathbf{u}, \mathbf{w} \in \mathbb{R}^{2d}$, let $L(\mathbf{u}, \mathbf{w}) =: A\mathbf{u} - \mathbf{w}$. Define $\mathcal{S}_A : \mathbb{R}^{2d} \times \mathbb{R}^{2d} \rightarrow \mathbb{C}$ by setting

$$\mathcal{S}_A(\mathbf{u}, \mathbf{w}) =: -L(\mathbf{u}, \mathbf{w})^t \left[\mathcal{P}_A + \frac{i}{2} \mathcal{R}_A \right] L(\mathbf{u}, \mathbf{w}) - i \omega_0(A\mathbf{u}, \mathbf{w}).$$

For example, when $A = O$ is unitary (that is, $P = I$) we have $\mathcal{P}_A = \frac{1}{2} I_d$, and

$$\mathcal{S}_A(\mathbf{u}, \mathbf{w}) =: -\frac{1}{2} \|A\mathbf{u} - \mathbf{w}\|^2 - i \omega_0(A\mathbf{u}, \mathbf{w}) = \psi_2(A\mathbf{u}, \mathbf{w}),$$

where ψ_2 is the universal exponent in the equivariant Szegő kernel asymptotics [SZ].

If R and S are unitary matrices, we have $\mathcal{S}_{RAS^t}(S\mathbf{u}, R\mathbf{w}) = \mathcal{S}_A(\mathbf{u}, \mathbf{w})$. Thus if $m_\tau =: \phi_{-\tau}^M(m)$, and $A = A_{\tau, m}$ represents $d_m \phi_{-\tau}^M : T_m M \rightarrow T_{m_\tau} M$, then \mathcal{S}_A does not depend on the choice of Heisenberg local coordinates, and is well-defined as a function

$$\mathcal{S}_{\tau, m} : T_m M \times T_{m_\tau} M \longrightarrow \mathbb{C}.$$

Similarly, $\nu : \mathbb{R} \times M \rightarrow \mathbb{R}$ given by

$$\nu(\tau, m) =: \sqrt{\det(Q_{A_{\tau, m}})} \tag{3}$$

is well-defined. If $d_m \phi_{-\tau}^M$ is unitary, $\nu(\tau, m) = 2^d$. Notice that, with $A = A_{\tau, m}$,

$$\nu(\tau, m) = \det(I + A^t A)^{1/2} = \det(A J_0 + J_0 A)^{1/2}. \tag{4}$$

As a further piece of notation, let $T_{\tau, m} \subseteq T_m M \times T_{m_\tau} M$ be the tangent space at (m, m_τ) to graph $(\phi_{-\tau}^M)$. In Heisenberg local coordinates, this is

$$T_{\tau, m} = \text{graph}(A) =: \{(\mathbf{u}, \mathbf{w}) \in \mathbb{C}^d \times \mathbb{C}^d : A\mathbf{u} = \mathbf{w}\}.$$

Finally, let $N_{\tau, m} =: T_{\tau, m}^\perp \subseteq T_m M \times T_{m_\tau} M$ be its orthocomplement for the Riemannian metric on $M \times M$.

Theorem 1.2. *Let R_τ be invariant zeroth order Toeplitz operators on X , with symbol $\varrho_\tau \in \mathcal{C}^\infty(M)$, and define U_τ by (1). Suppose $x \in X$, $x_\tau =: \phi_{-\tau}^X(x)$, $m =: \pi(x)$. Fix Heisenberg local coordinates centered at x and x_τ , respectively. Let $E > 0$ be a constant. Then, uniformly in $\mathbf{u} \in T_m M$ and $\mathbf{w} \in T_{m_\tau} M$ with $\|\mathbf{u}\|, \|\mathbf{w}\| \leq E k^{1/9}$ and $(\mathbf{u}, \mathbf{w}) \in N_{\tau, m}$, as $k \rightarrow +\infty$ we have*

$$U_{\tau, k} \left(x + \frac{\mathbf{u}}{\sqrt{k}}, x_\tau + \frac{\mathbf{w}}{\sqrt{k}} \right) \sim \varrho_\tau(m) \left(\frac{k}{\pi} \right)^d \frac{2^d}{\nu(\tau, m)} \cdot e^{\mathcal{S}_{\tau, m}(\mathbf{u}, \mathbf{w})} \cdot \left(1 + \sum_{j \geq 1} k^{-j/2} a_j(m, \tau, \mathbf{u}, \mathbf{w}) \right),$$

where $a_j(m, \tau, \mathbf{u}, \mathbf{w})$ is a polynomial in \mathbf{u} and \mathbf{w} , depending smoothly on m, τ .

In addition, the polynomial $a_j(m, \tau, X, Y)$ has the same parity as j .

Explicitly, the last claim is that $a_j(m, \tau, -X, -Y) = (-1)^j a_j(m, \tau, X, Y)$.

In particular, Theorem 1.2 describes an exponential decay of the rescaled kernel $U_{\tau, k} \left(x + \mathbf{u}/\sqrt{k}, x_\tau + \mathbf{w}/\sqrt{k} \right)$ along normal directions to the graph; the same will hold under the general transversality assumption $A\mathbf{u} \neq \mathbf{w}$.

Corollary 1.1. *If $U_{\tau, k}$ is unitary for $k \gg 0$, then*

$$|\varrho_\tau(m)| = 2^{-d/2} \sqrt{\nu(\tau, m)}. \quad (5)$$

The hypothesis in Corollary 1.1 means that U_τ is unitary, as an endomorphism of $H(X)$, on the complement of a finite dimensional subspace; it is obviously satisfied if U_τ itself is unitary.

We can give an analogue of the unitarization Lemma of [Z1]:

Corollary 1.2. *There exists a \mathcal{C}^∞ family R_τ of zeroth order Toeplitz operators R_τ such that if U_τ is defined by (1), then*

$$U_{\tau, k} \circ U_{\tau, k}^* = \Pi_k + O(k^{-\infty}), \quad U_{\tau, k}^* \circ U_{\tau, k} = \Pi_k + O(k^{-\infty}).$$

It follows from the proof of Corollary 1.2 that there is a canonical choice for R_τ , up to smoothing operators.

Remark 1.1. The functional argument on page 327 of [Z1] shows that U_τ may be modified so as to assume that it is actually unitary on the complement of a finite dimensional subspace of $H(X)$.

Remark 1.2. If $A = A_{\tau, m}$, by (4) we can rewrite the right hand side of (5) as

$$2^{-d/2} \det(A J_0 + J_0 A)^{1/4},$$

which tallies with the multiplier determined in §6 of [D] for the linear case.

Finally, to leading order U_τ satisfies the Shrödinger equation associated to f . Namely, let $D_\theta =: -i \partial / \partial \theta$, where $\partial / \partial \theta$ is the generator of the S^1 -action on X , and $\tilde{T}_f =: D_\theta \circ T_f$. Thus D_θ is the ‘number operator’ equal to $k \text{id}_{H(X)_k}$ on $H(X)_k$, and \tilde{T}_f is a self-adjoint invariant first order Toeplitz operator, and its restriction to $H(X)_k$ is $\tilde{T}_f^{(k)} = k T_f^{(k)}$.

Proposition 1.1. *In the situation of Theorem 1.2,*

$$\begin{aligned} & \left. \frac{d}{d\tau} U_{\tau,k} \right|_{\tau_0} \left(x + \frac{\mathbf{u}}{\sqrt{k}}, x_{\tau_0} + \frac{\mathbf{w}}{\sqrt{k}} \right) \\ &= \left(i \tilde{T}_f^{(k)} \circ U_{\tau_0,k} \right) \left(x + \frac{\mathbf{u}}{\sqrt{k}}, x_{\tau_0} + \frac{\mathbf{w}}{\sqrt{k}} \right) + e^{\mathcal{S}_{\tau_0,m}(\mathbf{u},\mathbf{w})} \cdot O(k^{d+1/2}). \end{aligned}$$

Under favorable conditions Theorems 1.1 and 1.2 also yield an asymptotic expansion for the trace of $U_{\tau,k}$. Let us consider the simplest case where $U_{\tau,k}$ only has isolated and non-degenerate fixed points m_1, \dots, m_r . Then on the one hand by Theorem 1.1

$$\text{trace}(U_{\tau,k}) = \int_X U_{\tau,k}(x, x) dV_X(x) \sim \sum_{i=1}^r \int_{X_i} \gamma_i(x) U_{\tau,k}(x, x) dV_X(x),$$

where γ_i is an invariant bump function, supported near $\pi^{-1}(x_i)$ and identically equal to one in a small neighborhood of $\pi^{-1}(x_i)$.

On the other hand, working in rescaled Heisenberg local coordinates centered at some x_i lying over m_i , and writing $\mathcal{S}_{\tau,m_i}(\mathbf{u}, \mathbf{u}) = -(1/2) \mathbf{u}^t S_i \mathbf{u}$ for a symmetric matrix S_i with positive real part, by Theorem 1.2 we have

$$\begin{aligned} & \int_{X_i} \gamma_i(x) U_{\tau,k}(x, x) dV_X(x) \\ & \sim k^{-d} \cdot \int_{\mathbb{C}^d} \gamma_i \left(x_i + \frac{\mathbf{u}}{\sqrt{k}} \right) U_{\tau,k} \left(x_i + \frac{\mathbf{u}}{\sqrt{k}}, x_i + \frac{\mathbf{u}}{\sqrt{k}} \right) d\mathbf{v} \\ & = k^{-d} \varrho_\tau(m_i) \left(\frac{k}{\pi} \right)^d \frac{2^d}{\nu(\tau, m_i)} \int_{\mathbb{C}^d} e^{-\frac{1}{2} \mathbf{u}^t S_i \mathbf{u}} d\mathbf{u} + \text{L.O.T.} \\ & = \frac{2^{2d}}{\nu(\tau, m_i)} \det(S_i)^{-1/2} + \text{L.O.T.} \end{aligned}$$

(here L.O.T. = lower order terms). Similar expansions may be obtained for higher dimensional symplectic fixed loci, adapting the arguments in [P1], but won’t be discussed here.

For the sake of simplicity, we have restricted our exposition to the complex projective setting; however, by the microlocal theory developed in [SZ], the previous results can be generalized to the case of almost complex symplectic manifolds.

2 Proof of Theorem 1.1.

Let $\Pi_\tau =: (\phi_{-\tau}^X)^* \circ \Pi$. In terms of Schwartz kernels,

$$\Pi_\tau = (\phi_{-\tau}^X \times \text{id}_X)^* (\Pi).$$

Then $U_\tau = R_\tau \circ \Pi_\tau$, and since R_τ and Π_τ are S^1 -invariant, they preserve each S^1 -equivariant summand $L^2(X)_k \subseteq L^2(X)$. Therefore, the restriction $U_{\tau,k} : H(X)_k \rightarrow H(X)_k$ is a composition $U_{\tau,k} = R_{\tau,k} \circ \Pi_{\tau,k}$, where $R_{\tau,k}$ and $\Pi_{\tau,k}$ are the restrictions of U_τ and R_τ . In fact, since R_τ and Π_τ commute with the orthogonal projection onto $L^2(X)_k$, we have

$$U_{\tau,k}(x, y) = R_{\tau,k} \circ \Pi_{\tau,k} = R_{\tau,k} \circ \Pi_\tau = R_\tau \circ \Pi_{\tau,k}. \quad (6)$$

Using distributional kernels, we can rewrite (6) in the form

$$\begin{aligned} U_{\tau,k}(x, y) &= \int_X R_{\tau,k}(x, z) \Pi_{\tau,k}(z, y) d\mu_X(z) \\ &= \int_X R_{\tau,k}(x, z) \Pi_k(\phi_{-\tau}^X(z), y) d\mu_X(z). \end{aligned} \quad (7)$$

where clearly $R_{\tau,k}, \Pi_{\tau,k} \in \mathcal{C}^\infty(X \times X)$. Furthermore, since Π is \mathcal{C}^∞ on $X \times X \setminus \text{diag}(X)$ by [?] [BS], so is R_τ ; therefore, $R_{\tau,k}(x, z) = O(k^{-\infty})$ as $k \rightarrow +\infty$, uniformly on the locus where $\text{dist}_X(x, S^1 \cdot z) \geq \delta$, for any fixed $\delta > 0$. Similarly, $\Pi_k(\phi_{-\tau}^X(z), y) = O(k^{-\infty})$ uniformly for $\text{dist}_X(y, S^1 \cdot \phi_{-\tau}^X(z)) \geq \delta$. It is well-known that $R_{\tau,k}(x, x) = \Pi_k(x, x) = O(k^d)$.

Lemma 2.1. *For any $\epsilon > 0$, we have $U_{\tau,k}(x, y) = O(k^{-\infty})$ as $k \rightarrow +\infty$, uniformly for $\text{dist}_X(y, \phi_{-\tau}^X(x)) \geq \epsilon$.*

Proof. Let $C, c > 0$ be such that for any $x_1, x_2 \in X$ we have

$$c \cdot \text{dist}_X(x_1, x_2) \leq \text{dist}_X(\phi_{-\tau}^X(x_1), \phi_{-\tau}^X(x_2)) \leq C \cdot \text{dist}_X(x_1, x_2), \quad (8)$$

and similarly for any $m_1, m_2 \in M$

$$c \cdot \text{dist}_M(m_1, m_2) \leq \text{dist}_M(\phi_{-\tau}^M(m_1), \phi_{-\tau}^M(m_2)) \leq C \cdot \text{dist}_M(m_1, m_2). \quad (9)$$

Choose $\epsilon > 0$ arbitrarily small, and suppose $\text{dist}_X(y, S^1 \cdot \phi_{-\tau}^X(x)) \geq \epsilon$. Define

$$\begin{aligned} V &=: \left\{ z \in X : \text{dist}_X(z, S^1 \cdot x) > \frac{\epsilon}{3C} \right\} = \left\{ z \in X : \text{dist}_M(\pi(z), \pi(x)) > \frac{\epsilon}{3C} \right\}, \\ W &=: \left\{ z \in X : \text{dist}_X(z, S^1 \cdot x) < \frac{\epsilon}{2C} \right\} = \left\{ z \in X : \text{dist}_M(\pi(z), \pi(x)) < \frac{\epsilon}{2C} \right\}. \end{aligned}$$

Then $\{V, W\}$ is an invariant open cover of X ; let $\{1 - \varrho, \varrho\}$ be an invariant partition of unity subordinate to it. We can rewrite (7) as

$$\begin{aligned} U_{\tau,k}(x, y) &= \int_V (1 - \varrho(z)) \cdot R_{\tau,k}(x, z) \Pi_k(\phi_{-\tau}^X(z), y) d\mu_X(z) \\ &\quad + \int_W \varrho(z) \cdot R_{\tau,k}(x, z) \Pi_k(\phi_{-\tau}^X(z), y) d\mu_X(z). \end{aligned} \quad (10)$$

Uniformly in $z \in V$, we have on the one hand $\Pi_k(\phi_{-\tau}^X(z), y) = O(k^d)$, and on the other $R_{\tau,k}(x, z) = O(k^{-\infty})$. Therefore, the first summand on the right hand side of (10) rapidly decreasing as $k \rightarrow +\infty$.

Uniformly in $z \in W$, we have on the one hand $R_{\tau,k}(x, z) = O(k^d)$, and on the other

$$\begin{aligned} \text{dist}_X(y, \phi_{-\tau}^X(z)) &\geq \text{dist}_X(y, \phi_{-\tau}^X(x)) - \text{dist}_X(\phi_{-\tau}^X(x), \phi_{-\tau}^X(z)) \quad (11) \\ &\geq \text{dist}_X(y, \phi_{-\tau}^X(x)) - C \cdot \text{dist}_X(x, z) > \frac{1}{2} \epsilon; \end{aligned}$$

therefore, $\Pi_k(\phi_{-\tau}^X(z), y) = O(k^{-\infty})$ on W , and the second summand is also rapidly decreasing. \square

We may thus assume that (x, y) lies in an arbitrary small invariant tubular neighborhood of $\widetilde{\text{graph}(\phi_{-\tau}^X)}$, that is, $\text{dist}_X(y, S^1 \cdot x_\tau) < \epsilon$ for some small $\epsilon > 0$, where $x_\tau =: \phi_{-\tau}^X(x)$. In view of (2), we may as well assume $\text{dist}_X(y, x_\tau) < \epsilon$, hence that $y = x_\tau + O(\epsilon)$ in any given system of local coordinates.

Let us define

$$\begin{aligned} V' &=: \left\{ z \in X : \text{dist}_X(z, S^1 \cdot x) > \frac{2}{c} \cdot \epsilon \right\}, \\ W' &=: \left\{ z \in X : \text{dist}_X(z, S^1 \cdot x) < \frac{3}{c} \cdot \epsilon \right\}, \end{aligned}$$

where c is as in (8). Let $\{1 - \varrho', \varrho'\}$ be an invariant partition of unity on X , subordinate to the open cover $\{V', W'\}$. In distributional short-hand, using the last equality in (7), we get

$$\begin{aligned} U_{\tau,k}(x, y) &= \int_{V'} (1 - \varrho'(z)) \cdot R_\tau(x, z) \Pi_k(\phi_{-\tau}^X(z), y) d\mu_X(z) \\ &\quad + \int_{W'} \varrho'(z) \cdot R_\tau(x, z) \Pi_k(\phi_{-\tau}^X(z), y) d\mu_X(z). \end{aligned} \quad (12)$$

Lemma 2.2. *The first integral on the right hand side of (12) is $O(k^{-\infty})$.*

Proof. On the S^1 -invariant open set $V' \subseteq X$, $R_\tau(x, \cdot)$ is \mathcal{C}^∞ and uniformly bounded. Furthermore, for $z \in V'$ we have

$$\begin{aligned} \text{dist}_X(\phi_{-\tau}^X(z), y) &\geq \text{dist}_X(\phi_{-\tau}^X(x), \phi_{-\tau}^X(z)) - \text{dist}_X(\phi_{-\tau}^X(x), y) \\ &\geq c \text{dist}_X(x, z) - \text{dist}_X(\phi_{-\tau}^X(x), y) > c \frac{2}{c} \cdot \epsilon - \epsilon = \epsilon. \end{aligned}$$

Therefore, $\Pi_k(\phi_{-\tau}^X(z), y) = O(k^{-\infty})$ uniformly for $z \in V'$. \square

It follows that as $k \rightarrow +\infty$

$$U_{\tau,k}(x, y) \sim \int_{W'} \varrho'(z) \cdot R_\tau(x, z) \Pi_k(\phi_{-\tau}^X(z), y) d\mu_X(z), \quad (13)$$

where \sim stands for 'equal asymptotics as'. Now in (13) z is in a small S^1 -invariant neighborhood of x , while $\phi_{-\tau}^X(z)$ and y are in a small S^1 -invariant neighborhood of $x_\tau = \phi_{-\tau}^X(x)$. Perhaps disregarding a smoothing term not contributing to the asymptotics, we may now introduce in (13) the microlocal descriptions of R_τ and Π as Fourier integral operators from [BS], and work in Heisenberg local coordinates centered at x and x_τ , respectively.

More precisely, by the discussion in [BS], [BSZ], [SZ] we have

$$\Pi(x', x'') =: \int_0^{+\infty} e^{iu\psi(x', x'')} s(u, x', x'') du \quad (14)$$

and

$$R_\tau(y', y'') =: \int_0^{+\infty} e^{it\psi(y', y'')} a_\tau(t, y', y'') dt; \quad (15)$$

here ψ is a complex phase of positive type, essentially determined by the Taylor expansion of the metric along the diagonal, and s, a_τ are semiclassical symbols. More precisely,

$$s(u, x', x'') \sim \sum_{j \geq 0} u^{d-j} s_j(x', x''), \quad a(t, x', x'') \sim \sum_{j \geq 0} t^{d-j} a_j(x', x''), \quad (16)$$

and since we are working in Heisenberg local coordinates centered at x and x_τ , respectively, we have

$$a_0(x, x) = \varrho_\tau(m) (k/\pi)^d, \quad s_0(x_\tau, x_\tau) = (k/\pi)^d. \quad (17)$$

Inserting (14) and (15) in (13), and performing the rescaling $t \mapsto kt$ and $u \mapsto ku$, we get

$$U_{\tau,k}(x, y) \sim \frac{k^2}{2\pi} \int_0^{+\infty} \int_0^\infty \int_{-\pi}^\pi \int_{W'} e^{ik\Psi_1} \mathcal{A}_1 dt du d\vartheta d\mu_X(z), \quad (18)$$

where

$$\Psi_1 =: t \psi(x, z) + u \psi(r_\vartheta(z_\tau), y) - \vartheta, \quad (19)$$

where $z_\tau =: \phi_{-\tau}^X(z)$, and

$$\mathcal{A}_1 =: a_\tau(kt, x, z) s(ku, r_\vartheta(z_\tau), y).$$

On the diagonal, we have $d_{(x,x)}\psi = (\alpha_x, -\alpha_x)$; more generally, $d_{(r_\theta(x),x)}\psi = (e^{i\theta}\alpha_{r_\theta(x)}, -e^{i\theta}\alpha_x)$. Working in Heisenberg local coordinates near x , we can write $z = x + (\theta, \mathbf{v})$, where $\|\mathbf{v}\| < (6/c)\epsilon$, say; consequently, in Heisenberg local coordinates near x_τ we have $z_\tau = x_\tau + (\theta, \mathbf{v}'_\tau)$, where again $\|\mathbf{v}'_\tau\| = O(\epsilon)$. In other words, $z = r_\theta(x) + O(\epsilon)$, $z_\tau = r_\theta(x_\tau) + O(\epsilon)$, and on the other hand $y = x_\tau + O(\epsilon)$. Therefore,

$$\partial_\theta \Psi_1 = t [-e^{-i\theta} + O(\epsilon)] + u [e^{i(\theta+\vartheta)} + O(\epsilon)], \quad (20)$$

$$\partial_\vartheta \Psi_1 = u [e^{i(\theta+\vartheta)} + O(\epsilon)] - 1. \quad (21)$$

It follows that

$$\|\nabla_{\theta,\vartheta}\Psi_1\| \geq \sqrt{(u-t)^2 + (u-1)^2} + O(\|(t,u)\| \cdot \epsilon). \quad (22)$$

Therefore, if ϵ is sufficiently small then $\|\nabla_{\theta,\vartheta}\Psi_1\|$ remains bounded away from zero when (u, t) does not belong to a small neighborhood of $(1, 1)$, and it is $\geq (1/2)\|(u, t)\|$, say, as $(u, t) \rightarrow \infty$.

We can rewrite (18) in local coordinates with $d\mu_X(z) = \mathcal{V}(\theta, \mathbf{v}) d\theta d\mathbf{v}$. In addition, θ and ϑ are really local coordinates on S^1 and therefore, upon introducing appropriate partitions of unity on the circle, the corresponding integration may be implicitly interpreted as compactly supported.

Integrating by parts in (θ, ϑ) we deduce from (22) that we only miss a negligible contribution to the asymptotics as $k \rightarrow +\infty$, if integration in (t, u) is restricted to a compact neighborhood of $(1, 1)$. Therefore,

Lemma 2.3. *Suppose $E \gg 0$, and let $\varrho_1 \in C_0^\infty((1/E, E))$ be ≥ 0 everywhere and $\equiv 1$ on $(2/E, E/2)$. Then*

$$U_{\tau,k}(x, y) \sim \frac{k^2}{2\pi} \int_{1/E}^E \int_{1/E}^E \int_{-\pi}^\pi \int_{W'} e^{ik\Psi_1} \mathcal{A}_2 dt du d\vartheta d\mu_X(z), \quad (23)$$

where $\mathcal{A}_2 =: \mathcal{A}_1 \cdot \varrho_1(t) \varrho_1(u)$.

Given $D > 0$ and with $C > 0$ as in (8), let us consider the invariant open sets

$$V'_k =: \left\{ z \in W' : \text{dist}_X(z, S^1 \cdot x) > \frac{D}{2C} \cdot k^{\varepsilon-1/2} \right\},$$

$$W'_k =: \left\{ z \in W' : \text{dist}_X(z, S^1 \cdot x) < \frac{2D}{3C} \cdot k^{\varepsilon-1/2} \right\},$$

and let $\{1 - \gamma_k, \gamma_k\}$ be an invariant partition of unity on X subordinate to it; we may assume that in local Heisenberg coordinates we have $\gamma_k(z) = \gamma_1(k^{1/2-\varepsilon} \|\mathbf{v}\|)$.

We can then rewrite (18) as follows

$$\begin{aligned} U_{\tau,k}(x, y) &\sim \frac{k^2}{2\pi} \int_{1/E}^E \int_{1/E}^E \int_{-\pi}^{\pi} \int_{V'_k} e^{ik\Psi_1} (1 - \gamma_k(z)) \mathcal{A}_1 dt du d\vartheta d\mu_X(z) \\ &\quad + \frac{k^2}{2\pi} \int_{1/E}^E \int_{1/E}^E \int_{-\pi}^{\pi} \int_{W'_k} e^{ik\Psi_1} \gamma_k(z) \mathcal{A}_1 dt du d\vartheta d\mu_X(z). \end{aligned} \quad (24)$$

Lemma 2.4. *The first summand on the right hand side of (24) is $O(k^{-\infty})$.*

Proof. For $z \in V'_k$, given (19) and by Corollary 1.3 of [BS] we have

$$|\partial_t \Psi_1| = |\psi(x, z)| \geq \Im \psi(x, z) \geq C_1 k^{2\varepsilon-1},$$

where $C_1 > 0$ is an appropriate constant. The statement follows by iteratively integrating by parts in dt . \square

Thus we are reduced to considering the second summand.

Lemma 2.5. *Given that $\text{dist}_X(y, S^1 \cdot x_\tau) > D k^{\varepsilon-1/2}$, the second summand on the right hand side of (24) is also $O(k^{-\infty})$.*

Proof. Setting $n =: \pi(y)$ and $m =: \pi(x)$, this may be rewritten $\text{dist}_M(n, m_\tau) > D k^{\varepsilon-1/2}$, where $m_\tau =: \phi_{-\tau}^M(m)$ and dist_M is the Riemannian distance on M . Similarly, if we set $p =: \pi(z)$ and $p_\tau =: \phi_{-\tau}^M(p)$ then for $z \in W'_k$ we have $\text{dist}_M(m, p) < (2D/3C) \cdot k^{\varepsilon-1/2}$, and therefore $\text{dist}_M(m_\tau, p_\tau) < (2D/3) \cdot k^{\varepsilon-1/2}$.

Therefore, for every $z \in W'_k$ and $\vartheta \in (-\pi, \pi)$, we have

$$\begin{aligned} \text{dist}_X(r_\vartheta(z_\tau), y) &\geq \text{dist}_M(p_\tau, n) \\ &\geq \text{dist}_M(n, m_\tau) - \text{dist}_M(m_\tau, p_\tau) > D k^{\varepsilon-1/2} - \frac{2D}{3} k^{\varepsilon-1/2} = \frac{D}{3} k^{\varepsilon-1/2}. \end{aligned}$$

We can now argue as in the proof of Lemma 2.4, and conclude that

$$|\partial_u \Psi_1| = |\psi(r_\vartheta(z_\tau), y)| \geq \Im \psi(r_\vartheta(z_\tau), y) \geq C_2 k^{2\varepsilon-1}.$$

Using this time using integration by parts in du , we conclude that the second summand on the right hand side of (24) is also $O(k^{-\infty})$ as $k \rightarrow +\infty$, if $\text{dist}_X(y, S^1 \cdot x_\tau) > D k^{\varepsilon-1/2}$. \square

Hence the left hand side of (24) is $O(k^{-\infty})$ for $k \rightarrow +\infty$, uniformly for $\text{dist}_X(y, S^1 \cdot x_\tau) > D k^{\varepsilon-1/2}$. This completes the proof of Theorem 1.1.

3 Proof of Theorem 1.2.

Let us set $\varepsilon = 1/9$ in the previous construction (this is just to fix ideas). In view of (24) and Lemma 2.5, writing $z = x + (\theta, \mathbf{v})$ in Heisenberg local coordinates we have

$$U_{\tau,k} \left(x + \frac{\mathbf{u}}{\sqrt{k}}, x_\tau + \frac{\mathbf{w}}{\sqrt{k}} \right) \sim \frac{k^2}{2\pi} \int_{1/E}^E \int_{1/E}^E \int_{-\pi}^\pi \int_{-\pi}^\pi \int_{\mathbb{C}^d} e^{ik\Psi'_1} \gamma_k(z) \mathcal{A}_1 \mathcal{V}(\theta, \mathbf{v}) dt du d\vartheta d\theta d\mathbf{v}, \quad (25)$$

where, recalling (19),

$$\begin{aligned} \Psi'_1 &= t \psi \left(x + \frac{\mathbf{u}}{\sqrt{k}}, x + (\theta, \mathbf{v}) \right) \\ &\quad + u \psi \left(\phi_{-\tau}^X(x + (\vartheta + \theta, \mathbf{v})), x_\tau + \frac{\mathbf{w}}{\sqrt{k}} \right) - \vartheta \end{aligned} \quad (26)$$

and integration in $d\mathbf{v}$ is over a ball centered at the origin and radius $O(\epsilon)$.

In particular, since Heisenberg local coordinates are isometric at the origin, again by Corollary 1.3 of [BS] for sufficiently small ϵ we have

$$|\partial_t \Psi_1| = \left| \psi \left(x + \frac{\mathbf{u}}{\sqrt{k}}, x + (\theta, \mathbf{v}) \right) \right| \geq a |\theta|^2,$$

for some constant $a > 0$. Integrating by parts in dt , as in Lemma 2.4, we conclude that only a small neighborhood of the origin in $(-\pi, \pi)$, say $(-\epsilon/2, \epsilon/2)$, gives a non-negligible contribution to the asymptotics.

Furthermore, by Lemma 3.2 of [P3] we have

$$\phi_{-\tau}^X(x + (\vartheta + \theta, \mathbf{v})) = x_\tau + \left(\vartheta + \theta + R_3(\mathbf{v}), A\mathbf{v} + R_2(\mathbf{v}) \right), \quad (27)$$

where R_j denotes a generic smooth function on an Euclidean space vanishing to j -th order at the origin (that is, $R_j(\mathbf{s}) = O(\|\mathbf{s}\|^j)$ for $\mathbf{s} \sim \mathbf{0}$). Therefore, if ϵ is small, $|\theta| < \epsilon/2$ and $|\vartheta| \geq \epsilon$ then

$$\begin{aligned} |\partial_u \Psi_1| &= \left| \psi \left(\phi_{-\tau}^X(x + (\vartheta + \theta, \mathbf{v})), x_\tau + \frac{\mathbf{w}}{\sqrt{k}} \right) \right| \\ &\geq a (\vartheta + \theta + R_3(\mathbf{v}))^2 \geq a' \epsilon^2, \end{aligned}$$

for some constant $a' > 0$. Thus integration by parts in du implies that the contribution to the asymptotics from the locus where $|\theta| < \epsilon/2$ and $|\vartheta| < \epsilon$ is also $O(k^{-\infty})$.

We can thus write

$$U_{\tau,k} \left(x + \frac{\mathbf{u}}{\sqrt{k}}, x_\tau + \frac{\mathbf{w}}{\sqrt{k}} \right) \sim \frac{k^2}{2\pi} \int_{1/E}^E \int_{1/E}^E \int_{-\epsilon}^\epsilon \int_{-\epsilon}^\epsilon \int_{\mathbb{C}^d} e^{ik\Psi'_1} \gamma_k(z) \mathcal{A}'_1 \mathcal{V}(\theta, \mathbf{v}) dt du d\vartheta d\theta d\mathbf{v}, \quad (28)$$

where $\mathcal{A}'_1 =: \varrho(\theta, \vartheta) \mathcal{A}_1$, and $\varrho(\theta, \vartheta)$ is an appropriate bump function on \mathbb{R}^2 , $\equiv 1$ near the origin and supported on a ball of radius $O(\epsilon)$.

Let us now operate the rescaling $\mathbf{v} \mapsto \mathbf{v}/\sqrt{k}$; we get

$$U_{\tau,k} \left(x + \frac{\mathbf{u}}{\sqrt{k}}, x_\tau + \frac{\mathbf{w}}{\sqrt{k}} \right) \sim \frac{k^{2-d}}{2\pi} \int_{\mathbb{C}^d} \left[\int_{1/E}^E \int_{1/E}^E \int_{-\epsilon}^\epsilon \int_{-\epsilon}^\epsilon e^{ik\Psi_2} \mathcal{A}_2 \cdot \mathcal{V} \left(\theta, \frac{\mathbf{v}}{\sqrt{k}} \right) dt du d\vartheta d\theta \right] d\mathbf{v}. \quad (29)$$

where $\Psi_2 = \Psi'_1(\mathbf{u}, \mathbf{w}, \mathbf{v}, \theta)$ is Ψ'_1 expressed in rescaled Heisenberg coordinates, and similarly for $\mathcal{A}_2 =: \gamma(k^{-1/9}\mathbf{v}) \mathcal{A}'_1$ (dependence on τ and k is omitted). Integration in $d\mathbf{v}$ is now over a ball centered at the origin and of radius $O(k^{1/9})$ in \mathbb{C}^d .

Thus, by (26),

$$\begin{aligned} \Psi_2 &= t \psi \left(x + \frac{\mathbf{u}}{\sqrt{k}}, x + \left(\theta, \frac{\mathbf{v}}{\sqrt{k}} \right) \right) \\ &\quad + u \psi \left(\phi_{-\tau}^X \left(x + \left(\vartheta + \theta, \frac{\mathbf{v}}{\sqrt{k}} \right) \right), x_\tau + \frac{\mathbf{w}}{\sqrt{k}} \right) - \vartheta. \end{aligned} \quad (30)$$

Let R_j denote a generic smooth function on an Euclidean space vanishing to j -th order at the origin (that is, $R_j(\mathbf{s}) = O(\|\mathbf{s}\|^j)$ for $\mathbf{s} \sim \mathbf{0}$). By the discussion in §3 of [SZ], we have

$$\begin{aligned} &t \psi \left(x + \frac{\mathbf{u}}{\sqrt{k}}, x + \left(\theta, \frac{\mathbf{v}}{\sqrt{k}} \right) \right) \\ &= it [1 - e^{-i\theta}] - \frac{it}{k} \psi_2(\mathbf{u}, \mathbf{v}) e^{-i\theta} + t R_3 \left(\frac{\mathbf{u}}{\sqrt{k}}, \frac{\mathbf{v}}{\sqrt{k}} \right) e^{-i\theta}, \end{aligned} \quad (31)$$

where

$$\psi_2(\mathbf{u}, \mathbf{v}) =: -i\omega_0(\mathbf{u}, \mathbf{v}) - \frac{1}{2} \|\mathbf{u} - \mathbf{v}\|^2. \quad (32)$$

Again by Lemma 3.2 of [P3], we have

$$\begin{aligned} &\phi_{-\tau}^X \left(x + \left(\vartheta + \theta, \frac{\mathbf{v}}{\sqrt{k}} \right) \right) \\ &= x_\tau + \left(\vartheta + \theta + R_3 \left(\frac{\mathbf{v}}{\sqrt{k}} \right), \frac{A\mathbf{v}}{\sqrt{k}} + R_2 \left(\frac{\mathbf{v}}{\sqrt{k}} \right) \right). \end{aligned} \quad (33)$$

It follows that

$$\begin{aligned} & u \psi \left(\phi_{-\tau}^X \left(x + \left(\vartheta + \theta, \frac{\mathbf{v}}{\sqrt{k}} \right) \right), x + \frac{\mathbf{w}}{\sqrt{k}} \right) \\ &= iu [1 - e^{i(\theta+\vartheta)}] - \frac{i u}{k} \psi_2(A\mathbf{v}, \mathbf{w}) e^{i(\theta+\vartheta)} + u R_3 \left(\frac{A\mathbf{v}}{\sqrt{k}}, \frac{\mathbf{w}}{\sqrt{k}} \right) e^{i(\theta+\vartheta)}, \end{aligned} \quad (34)$$

Inserting (31) and (34) in (30), we can rewrite (29) as follows:

$$\begin{aligned} & U_{\tau,k} \left(x + \frac{\mathbf{u}}{\sqrt{k}}, x_\tau + \frac{\mathbf{w}}{\sqrt{k}} \right) \\ & \sim \frac{k^{2-d}}{2\pi} \int_{\mathbb{C}^d} \left[\int_{1/E}^E \int_{1/E}^E \int_{-\epsilon}^\epsilon \int_{-\epsilon}^\epsilon e^{i k \Psi} \mathcal{B}_k \cdot \mathcal{V} \left(\theta, \frac{\mathbf{v}}{\sqrt{k}} \right) dt du d\vartheta d\theta \right] d\mathbf{v}, \end{aligned} \quad (35)$$

where

$$\Psi =: i t [1 - e^{-i\theta}] + i u [1 - e^{i(\theta+\vartheta)}] - \vartheta, \quad (36)$$

$$\begin{aligned} \mathcal{B}_k &=: \exp \left(t \psi_2(\mathbf{u}, \mathbf{v}) e^{-i\theta} + u \psi_2(A\mathbf{v}, \mathbf{w}) e^{i(\theta+\vartheta)} \right) \\ & \exp \left(i k t R_3 \left(\frac{\mathbf{u}}{\sqrt{k}}, \frac{\mathbf{v}}{\sqrt{k}} \right) e^{-i\theta} + i k u R_3 \left(\frac{A\mathbf{v}}{\sqrt{k}}, \frac{\mathbf{w}}{\sqrt{k}} \right) e^{i(\theta+\vartheta)} \right) \cdot \mathcal{A}_2. \end{aligned} \quad (37)$$

Lemma 3.1. *There exists $a = a_\tau > 0$ such that for any $(\mathbf{u}, \mathbf{w}) \in N_{\tau,m}$ and $\mathbf{v} \in T_m M$ we have*

$$\Re \left(t \psi_2(\mathbf{u}, \mathbf{v}) e^{-i\theta} + u \psi_2(A\mathbf{v}, \mathbf{w}) e^{i(\theta+\vartheta)} \right) \leq -a (\|\mathbf{u}\|^2 + \|\mathbf{w}\|^2 + \|\mathbf{v}\|^2),$$

Proof. The linear map $N_{\tau,m} \times T_m M \rightarrow T_m M \times T_{m_\tau} M$ given in local coordinates by $(\mathbf{u}, \mathbf{w}, \mathbf{v}) \mapsto (\mathbf{u} - \mathbf{v}, A\mathbf{v} - \mathbf{w})$ is injective by assumption. Therefore, $\|\mathbf{u} - \mathbf{v}\|^2 + \|A\mathbf{v} - \mathbf{w}\|^2 \geq C (\|\mathbf{u}\|^2 + \|\mathbf{w}\|^2 + \|\mathbf{v}\|^2)$ for some $C > 0$. The statement follows from the definition of ψ_2 and the fact that $|\theta|, |\vartheta| < \epsilon$. \square

The second exponent on the right hand side of (37), on the other hand, is bounded for $\|\mathbf{u}\|, \|\mathbf{w}\|, \|\mathbf{v}\| = O(k^{1/9})$. Taylor expanding the exponent at the origin yields an asymptotic expansion of the corresponding exponential in descending powers of $k^{-1/2}$, which may be incorporated into the amplitude.

We are then in a position to apply the stationary phase Lemma, regarding the inner integral in (35) as an oscillatory integral, with phase $\Psi = \Psi(t, \theta, u, \vartheta)$ having non-negative imaginary part. A straightforward computation yields:

Lemma 3.2. *Ψ has the unique stationary point*

$$(t_0, \theta_0, u_0, \vartheta_0) = (1, 0, 1, 0).$$

Furthermore, the Hessian matrix there is

$$H(\Psi)_0 = \begin{pmatrix} 0 & -1 & 0 & 0 \\ -1 & 2i & 1 & i \\ 0 & 1 & 0 & 1 \\ 0 & i & 1 & i \end{pmatrix}.$$

In particular, $\det(H(\Psi)_0) = 1$ and the stationary point is non-degenerate.

In addition, $H(\Psi)_0 = H(1)$, where for $0 \leq s \leq 1$ we set

$$H(s) =: \begin{pmatrix} 0 & -1 & 0 & 0 \\ -1 & 2si & 1 & si \\ 0 & 1 & 0 & 1 \\ 0 & si & 1 & si \end{pmatrix}.$$

We have $H(s) = 1$ for every s , and $H(0)$ is real and symmetric with vanishing signature. Therefore,

$$\sqrt{\det\left(\frac{k H(\Psi)_0}{2\pi i}\right)} = \left(\frac{k}{2\pi}\right)^2. \quad (38)$$

Applying the stationary phase Lemma, we get for the inner integral in (35) an asymptotic expansion in descending powers of $k^{-1/2}$, and it follows from Lemma 3.1 and the bound on the N -th step remainder that this expansion may be integrated term by term in $d\mathbf{v}$, yielding an asymptotic expansion for (35). Given (17) and (38), and since $\mathcal{V}(\theta, \mathbf{0}) = 1/(2\pi)$, the leading term is

$$\frac{k^d}{\pi^{2d}} \varrho_\tau(m) \int_{\mathbb{C}^d} e^{\psi_2(\mathbf{u}, \mathbf{v}) + \psi_2(A\mathbf{v}, \mathbf{w})} d\mathbf{v}. \quad (39)$$

With the change of variable $\mathbf{v} = \mathbf{r} + \mathbf{u}$, we get

$$\begin{aligned} & \psi_2(\mathbf{u}, \mathbf{v}) + \psi_2(A\mathbf{v}, \mathbf{w}) \\ &= \psi_2(A\mathbf{u}, \mathbf{w}) - i\omega_0 (A^{-1}L(\mathbf{u}, \mathbf{w}), \mathbf{r}) - \mathbf{r}^t A^t L(\mathbf{u}, \mathbf{w}) - \frac{1}{2} \mathbf{r}^t Q \mathbf{r}, \end{aligned} \quad (40)$$

where $L = L_A$ and $Q = Q_A$ are as in Definition 1.2. With the further replacement $\mathbf{r} = \mathbf{s} - Q^{-1}AL(\mathbf{u}, \mathbf{w})$ in (40), we get

$$\begin{aligned} \psi_2(\mathbf{u}, \mathbf{v}) + \psi_2(A\mathbf{v}, \mathbf{w}) &= \Gamma(\mathbf{u}, \mathbf{w}) \\ &\quad - i \mathbf{s}^t J_0 A^{-1} L(\mathbf{u}, \mathbf{w}) - \frac{1}{2} \mathbf{s}^t Q \mathbf{s}, \end{aligned} \quad (41)$$

where

$$\begin{aligned}\Gamma(\mathbf{u}, \mathbf{w}) &=: \psi_2(A\mathbf{u}, \mathbf{w}) + i\omega_0(A^{-1}L(\mathbf{u}, \mathbf{w}), Q^{-1}A^tL(\mathbf{u}, \mathbf{w})) \\ &\quad + \frac{1}{2}L(\mathbf{u}, \mathbf{w})^t A Q^{-1} A^t L(\mathbf{u}, \mathbf{w}).\end{aligned}\quad (42)$$

Therefore, the leading term (39) is

$$\frac{k^d}{\pi^{2d}} \varrho_\tau(m) e^{\Gamma(\mathbf{u}, \mathbf{w})} \int_{\mathbb{C}^d} e^{-i\mathbf{s}^t J_0 A^{-1} L(\mathbf{u}, \mathbf{w}) - \frac{1}{2}\mathbf{s}^t Q \mathbf{s}} d\mathbf{s}. \quad (43)$$

Let us set

$$F(\mathbf{u}, \mathbf{w}) =: -J_0 A^{-1} L(\mathbf{u}, \mathbf{w}) = -A^t J_0 L(\mathbf{u}, \mathbf{w}), \quad G(\mathbf{u}, \mathbf{w}) =: A^t L(\mathbf{u}, \mathbf{w}). \quad (44)$$

Then with some manipulations (43) is

$$\begin{aligned}&\left(\frac{k}{\pi}\right)^d \varrho_\tau(m) \cdot \frac{2^d}{\sqrt{\det(Q)}} \cdot e^{\Gamma(\mathbf{u}, \mathbf{w}) - \frac{1}{2}F(\mathbf{u}, \mathbf{w})^t Q^{-1} F(\mathbf{u}, \mathbf{w})} \\ &= \left(\frac{k}{\pi}\right)^d \varrho_\tau(m) \cdot \frac{2^d}{\sqrt{\det(Q)}} \cdot e^{S(\mathbf{u}, \mathbf{w})},\end{aligned}\quad (45)$$

where

$$\begin{aligned}S(\mathbf{u}, \mathbf{w}) &=: \psi_2(A\mathbf{u}, \mathbf{w}) - i G(\mathbf{u}, \mathbf{w}) Q^{-1} F(\mathbf{u}, \mathbf{w}) \\ &\quad + \frac{1}{2} G(\mathbf{u}, \mathbf{w})^t Q^{-1} G(\mathbf{u}, \mathbf{w}) - \frac{1}{2} F(\mathbf{u}, \mathbf{w})^t Q^{-1} F(\mathbf{u}, \mathbf{w}).\end{aligned}\quad (46)$$

Lemma 3.3. *If $\mathcal{P} = \mathcal{P}_A$ and $\mathcal{R} = \mathcal{R}_A$ are as in Definition 1.2, then*

$$S(\mathbf{u}, \mathbf{w}) = -L(\mathbf{u}, \mathbf{w})^t \left(\mathcal{P} + \frac{i}{2} \mathcal{R} \right) L(\mathbf{u}, \mathbf{w}) - i\omega_0(A\mathbf{u}, \mathbf{w}).$$

Proof. By (32), $\psi_2(A\mathbf{u}, \mathbf{w}) = -i\omega_0(A\mathbf{u}, \mathbf{w}) - (1/2) \|L(\mathbf{u}, \mathbf{w})\|^2$. Using this and (44) in (46), we get

$$\begin{aligned}S(\mathbf{u}, \mathbf{w}) &= -\frac{1}{2} L(\mathbf{u}, \mathbf{w})^t [I - J_0 A Q^{-1} A^t J_0 - A Q^{-1} A^t] L(\mathbf{u}, \mathbf{w}) \\ &\quad + i L(\mathbf{u}, \mathbf{w})^t A Q^{-1} A^t J_0 L(\mathbf{u}, \mathbf{w}) - i\omega_0(A\mathbf{u}, \mathbf{w}).\end{aligned}\quad (47)$$

Writing $A = OP$, we get (see Lemma 2.1 of [?])

$$I - J_0 A Q^{-1} A^t J_0 - A Q^{-1} A^t = 2 O Q^{-1} O^t = 2 \mathcal{P}, \quad (48)$$

and on the other hand $AQ^{-1}A^t J_0 = OP^2Q^{-1}J_0O^t$; on the other hand, since P is symplectic and symmetric, $(I + P^2) J_0 = J_0 (I + P^{-2})$. Therefore,

$$\begin{aligned} AQ^{-1}A^t J_0 + (AQ^{-1}A^t J_0)^t &= O[P^2 - I] Q^{-1}J_0O^t \\ &= -\mathcal{R}. \end{aligned} \quad (49)$$

The statement follows by inserting (48) and (49) in (47). \square

Thus $S(\mathbf{u}, \mathbf{w}) = \mathcal{S}_A(\mathbf{u}, \mathbf{w})$, and this proves that the leading term of the asymptotic expansion is as claimed in the statement of the Theorem.

By the same arguments, the general lower order term in the expansion has the form

$$\begin{aligned} k^{d-j/2} \varrho_\tau(m) \int_{\mathbb{C}^d} P_j(\mathbf{u}, \mathbf{w}, \mathbf{v}) e^{\psi_2(\mathbf{u}, \mathbf{v}) + \psi_2(A\mathbf{v}, \mathbf{w})} d\mathbf{v} \\ = k^{d-j/2} \varrho_\tau(m) e^{\Gamma(\mathbf{u}, \mathbf{w})} \int_{\mathbb{C}^d} e^{-i\mathbf{s}^t J_0 A^{-1} L(\mathbf{u}, \mathbf{w})} \tilde{P}_j(\mathbf{u}, \mathbf{w}, D) \left(e^{-\frac{1}{2}\mathbf{s}^t Q \mathbf{s}} \right) d\mathbf{s}, \end{aligned} \quad (50)$$

where j is a positive integer, P_j a polynomial, and \tilde{P}_j a differential operator with coefficients depending polynomially on (\mathbf{u}, \mathbf{w}) . This may be rewritten

$$k^{d-j/2} \varrho_\tau(m) e^{\mathcal{S}_{\tau, m}(\mathbf{u}, \mathbf{w})} \cdot a_j(m, \tau, \mathbf{u}, \mathbf{w}),$$

for a certain polynomial a_j , depending smoothly on m and τ .

Let us now consider the last claim of the Theorem. Since on the one hand the asymptotic expansions for the amplitudes in (16) go down by integer steps, and on the other the inner integral in (35) is oscillatory in k , the appearance of half-integer powers of k is the asymptotic expansion of Theorem 1.2 originates solely from Taylor expanding the amplitude in (37) in the rescaled arguments \mathbf{u}/\sqrt{k} , \mathbf{w}/\sqrt{k} , \mathbf{v}/\sqrt{k} . Therefore, the general term (50) of the expansion is actually a sum of terms of the form

$$k^{r-|\ell|/2} \varrho_\tau(m) \int_{\mathbb{C}^d} P_\ell(\mathbf{u}, \mathbf{w}, \mathbf{v}) e^{\psi_2(\mathbf{u}, \mathbf{v}) + \psi_2(A\mathbf{v}, \mathbf{w})} d\mathbf{v}, \quad (51)$$

where r is an integer, $P_\ell(\mathbf{u}, \mathbf{w}, \mathbf{v})$ is a polyhomogenous polynomial in $(\mathbf{u}, \mathbf{w}, \mathbf{v})$, of polydegree $\ell = (l_{\mathbf{u}}, l_{\mathbf{w}}, l_{\mathbf{v}})$, and $|\ell| = l_{\mathbf{u}} + l_{\mathbf{w}} + l_{\mathbf{v}}$; the coefficients are smooth in m .

By the previous passages, involving the change of variable $\mathbf{v} = \mathbf{s} - Q^{-1}AL(\mathbf{u}, \mathbf{w}) + \mathbf{u}$, the integral in (51) may be rewritten as a sum of terms of the form

$$e^{\Gamma(\mathbf{u}, \mathbf{w})} \int_{\mathbb{C}^d} e^{-i\mathbf{s}^t J_0 A^{-1} L(\mathbf{u}, \mathbf{w})} R_{\ell''}(\mathbf{u}, \mathbf{w}, \mathbf{s}) e^{-\frac{1}{2}\mathbf{s}^t Q \mathbf{s}} d\mathbf{s}, \quad (52)$$

where again $R_{\ell'}$ is polyhomogenous, of polydegree ℓ' with $|\ell'| = |\ell|$.

In turn, (52) splits as a sum of terms of the form

$$\begin{aligned} & e^{\Gamma(\mathbf{u}, \mathbf{w})} \int_{\mathbb{C}^d} e^{-i \mathbf{s}^t J_0 A^{-1} L(\mathbf{u}, \mathbf{w})} \widehat{R}_{\ell'}(\mathbf{u}, \mathbf{w}, D_{\mathbf{s}}) \left(e^{-\frac{1}{2} \mathbf{s}^t Q \mathbf{s}} \right) d\mathbf{s} \\ &= c \widehat{R}_{\ell'}(\mathbf{u}, \mathbf{w}, F(\mathbf{u}, \mathbf{w})) e^{\mathcal{S}_{\tau, m}(\mathbf{u}, \mathbf{w})} \end{aligned} \quad (53)$$

where now $\widehat{R}_{\ell''}$ is polyhomogenous of polydegree $\ell'' = (l'_{\mathbf{u}}, l'_{\mathbf{w}}, 2a + l'_{\mathbf{s}})$ for some integer a .

Summing up, the general summand (51) splits as a linear combination of terms of the form

$$k^{b - |\ell'''|/2} \cdot \widetilde{R}_{\ell'''}(\mathbf{u}, \mathbf{w}) e^{\mathcal{S}_{\tau, m}(\mathbf{u}, \mathbf{w})},$$

where b is an integer, and $\widetilde{R}_{\ell'''}$ is polyhomogenous of polydegree $\ell''' = (l'''_{\mathbf{u}}, l'''_{\mathbf{w}})$. The claim follows.

4 Proof of Corollary 1.1

If $U_{\tau, k}$ is unitary for $k \gg 0$, then $U_{\tau, k} \circ U_{\tau, k}^* = \Pi_k$ for k large. In particular, for any $x \in X$ this implies

$$(U_{\tau, k} \circ U_{\tau, k}^*)(x, x) = \Pi_k(x, x) = \left(\frac{k}{\pi} \right)^d + O(k^{d-1}). \quad (54)$$

Now we have

$$(U_{\tau, k} \circ U_{\tau, k}^*)(x, x) = \int_X U_{\tau, k}(x, y) U_{\tau, k}^*(y, x) d\mu_X(y), \quad (55)$$

where $U_{\tau, k}^*(y, x) = \overline{U_{\tau, k}(x, y)}$.

By Theorem 1.1 with $\varepsilon = 1/9$, only a shrinking S^1 -invariant neighborhood of x_{τ} , of radius say $O(k^{-7/18})$, contributes non-negligibly to the asymptotics. So introducing Heisenberg local coordinates centered at x_{τ} , and with γ_k as in (24), we can rewrite (55) as follows:

$$\begin{aligned} (U_{\tau, k} \circ U_{\tau, k}^*)(x, x) &\sim \int_X U_{\tau, k}(x, y) U_{\tau, k}^*(y, x) \gamma_k(y) d\mu_X(y) \\ &= k^{-d} \int_X \left| U_{\tau, k} \left(x, x_{\tau} + \left(\theta, \frac{\mathbf{v}}{\sqrt{k}} \right) \right) \right|^2 \mathcal{V} \left(\theta, \frac{\mathbf{v}}{\sqrt{k}} \right) \gamma(k^{-1/9} \mathbf{v}) d\mathbf{v} d\theta. \end{aligned} \quad (56)$$

Using Theorem 1.2, and recalling that $\mathcal{V}(\theta, \mathbf{0}) = 1/(2\pi)$, we get with $Q = Q_A$:

$$\begin{aligned} (U_{\tau, k} \circ U_{\tau, k}^*)(x, x) &\sim \frac{k^d}{\pi^{2d}} |\varrho_{\tau}(m)|^2 \frac{2^{2d}}{\det(Q)} \int_{\mathbb{C}^d} e^{2\Re(S(\mathbf{0}, \mathbf{v}))} d\mathbf{v} \\ &\quad + O(k^{d-1}); \end{aligned} \quad (57)$$

in passing, that the remainder is $O(k^{d-1})$ rather than $O(k^{d-1/2})$ follows directly from the parity Claim in Theorem 1.2. In view of Definition 1.2,

$$2 \Re(\mathcal{S}(\mathbf{0}, \mathbf{v})) = -2 \mathbf{v}^t \mathcal{P} \mathbf{v} = -2 \mathbf{v}^t O Q^{-1} O^t \mathbf{v}. \quad (58)$$

Setting $\mathbf{s} = O^t \mathbf{v}$, and then $\mathbf{r} = \mathbf{s}/2$, the integral in (57) is

$$\begin{aligned} \int_{\mathbb{C}^d} e^{-2 \mathbf{v}^t O Q^{-1} O^t \mathbf{v}} d\mathbf{v} &= \int_{\mathbb{C}^d} e^{-2 \mathbf{s}^t Q^{-1} \mathbf{s}} d\mathbf{s} \\ &= 2^{-2d} \int_{\mathbb{C}^d} e^{-\frac{1}{2} \mathbf{r}^t Q^{-1} \mathbf{r}} d\mathbf{r} = 2^{-2d} (2\pi)^d \sqrt{\det(Q)}. \end{aligned} \quad (59)$$

Inserting (59) in (57), we get

$$(U_{\tau,k} \circ U_{\tau,k}^*)(x, x) = \left(\frac{k}{\pi}\right)^d |\varrho_\tau(m)|^2 \frac{2^d}{\sqrt{\det(Q)}} + O(k^{d-1}). \quad (60)$$

Comparing (60) with (54), we conclude that $|\varrho_\tau(m)| = 2^{-d/2} \cdot \det(Q)^{1/4}$ if $U_{\tau,k}$ is unitary for $k \gg 0$.

5 Proof of Corollary 1.2.

As a preliminary remark, we recall that for any integer $a \geq 0$ a Toeplitz operator Q of degree $-a$ may be written microlocally in the form

$$Q(y', y'') =: \int_0^{+\infty} e^{it\psi(y', y'')} q(t, y', y'') dt, \quad (61)$$

where the amplitude q is a semiclassical symbol admitting an asymptotic expansion of the form

$$q(t, y', y'') \sim \sum_{j \geq a} t^{d-j} q_j(y', y''). \quad (62)$$

On the other hand, if Q is S^1 -invariant then by the discussion in [?] it also admits an asymptotic expansion of the form

$$Q \sim \sum_{j \geq a} T^{-j} \Pi \circ M_{f_j} \circ \Pi, \quad (63)$$

where now $f_j \in \mathcal{C}^\infty(M)$ is implicitly pulled-back to X , M_{f_j} is multiplication by f_j , and T is a parametrix (in the Toeplitz sense) of the elliptic first order Toeplitz operator associated to the generator of the structure circle action.

The symbol of Q , in particular, is the function $\sigma(Q) : \Sigma \rightarrow \mathbb{C}$ given by $\sigma(Q)(x, r\alpha_x) = r^{-a} f_a(m)$, where $m =: \pi(x)$.

When working in Heisenberg local coordinates centered at $x \in X$,

$$q_a(x, x) = \frac{1}{\pi^d} f_a(m). \quad (64)$$

Now consider the intrinsically defined asymptotic expansion

$$\Pi_k(x, x) \sim \left(\frac{k}{\pi}\right)^d + \sum_{j \geq 1} k^{d-j} a_j(m), \quad (65)$$

for certain $a_j \in \mathcal{C}^\infty(M)$. Let $U_\tau = U_\tau^{[1]}$ be as in (1), for some zeroth order Toeplitz operator $R_\tau = R_\tau^{[1]}$ with $\varrho_\tau(m) = 2^{-d/2} \sqrt{\nu(\tau, m)}$. The proof of Corollary 1.1 implies

$$\left(U_{\tau,k}^{[1]} \circ (U_{\tau,k}^{[1]})^*\right)(x, x) \sim \left(\frac{k}{\pi}\right)^d + \sum_{j \geq 1} k^{d-j} a_j^{[1]}(m), \quad (66)$$

for certain $a_j^{[1]} \in \mathcal{C}^\infty(M)$; that the expansion goes down by integer steps can be seen - for instance - by using in (56) the parity properties of the a_j 's asserted in Theorem 1.2.

Next let $U_\tau^{[2]}$ be again as in (1), but with $R_\tau^{[1]}$ replaced by $R_\tau^{[2]} =: R_\tau^{[1]} + \Sigma_\tau^{[1]}$, where

$$\Sigma_\tau^{[1]} = T^{-1} \Pi \circ M_{f_1} \circ \Pi,$$

for a suitable $f_1 \in \mathcal{C}^\infty(M \times \mathbb{R})$. Thus $\Sigma_\tau^{[1]}$ is a Toeplitz operator of degree -1 , hence microlocally of the form

$$\Sigma_\tau^{[1]}(y', y'') =: \int_0^{+\infty} e^{it\psi(y', y'')} \sigma_\tau^{[1]}(t, y', y'') dt, \quad (67)$$

with

$$\sigma_\tau^{[1]}(t, y', y'') \sim \sum_{j \geq 1} t^{d-j} \sigma_{\tau j}^{[1]}(y', y''),$$

and $\sigma_{\tau 1}^{[1]}(x, x) = f_1(m, \tau)/\pi^d$. Applying the stationary phase argument in the proof of Theorem 1.2, and arguing as for (60), we get

$$\begin{aligned} & \left(U_{\tau,k}^{[2]} \circ (U_{\tau,k}^{[2]})^*\right)(x, x) \\ & \sim \left(U_{\tau,k}^{[1]} \circ (U_{\tau,k}^{[1]})^*\right)(x, x) + \frac{k^{d-1}}{\pi^d} \varrho_\tau(m) \cdot 2 \Re(f_1(m, \tau)) \frac{2^d}{\sqrt{\det(Q)}} \\ & \quad + O(k^{d-2}). \end{aligned} \quad (68)$$

It is then clear that $f_1 : M \rightarrow \mathbb{R}$ may be chosen uniquely so that (68) agrees with (65) up to $O(k^{d-2})$.

Proceeding inductively, there are unique real $f_j \in \mathcal{C}^\infty(M \times \mathbb{R})$, such that if $R_\tau^{[\infty]} \sim R_\tau + \sum_{j \geq 1} T^{-j} \Pi \circ M_{f_j} \circ \Pi$ and $U_\tau^{[\infty]}$ is as in (1), with $R_\tau^{[\infty]}$ in place of R_τ , then

$$\left(U_{\tau,k}^{[\infty]} \circ (U_{\tau,k}^{[\infty]})^* \right) (x, x) \sim \Pi_k(x, x), \quad (69)$$

hence $U_{\tau,k}^{(\infty)} \circ (U_{\tau,k}^{(\infty)})^* = \Pi_k + O(k^{-\infty})$. Therefore, $(U_{\tau,k}^{(\infty)})^* \circ U_{\tau,k}^{(\infty)} \geq 0$ and $(U_{\tau,k}^{(\infty)})^* \circ U_{\tau,k}^{(\infty)} = \left((U_{\tau,k}^{(\infty)})^* \circ U_{\tau,k}^{(\infty)} \right)^2 + O(k^{-\infty})$; working in an orthonormal basis of eigenvectors, we conclude that $(U_{\tau,k}^{(\infty)})^* \circ U_{\tau,k}^{(\infty)} = \Pi_k + O(k^{-\infty})$ as well.

6 Proof of Proposition 1.1

The proof is an adaptation of the one for Theorem 1.2, so we'll be rather sketchy. Working at a fixed τ_0 , let us define operators $\tilde{U}_{\tau,k} =: U_{\tau_0+\tau/\sqrt{k},k}$, so that

$$\left. \frac{dU_{\tau,k}}{d\tau} \right|_{\tau_0} = \sqrt{k} \cdot \left. \frac{d\tilde{U}_{\tau,k}}{d\tau} \right|_0. \quad (70)$$

Arguing as in the proof of Theorem 1.2, we have in place of (29)

$$\begin{aligned} & \tilde{U}_{\tau,k} \left(x + \frac{\mathbf{u}}{\sqrt{k}}, x_{\tau_0} + \frac{\mathbf{w}}{\sqrt{k}} \right) \\ & \sim \frac{k^{2-d}}{2\pi} \int_{\mathbb{C}^d} \left[\int_{1/E}^E \int_{1/E}^E \int_{-\epsilon}^\epsilon \int_{-\epsilon}^\epsilon e^{ik\tilde{\Psi}_2} \tilde{\mathcal{A}}_2 \cdot \nu \left(\theta, \frac{\mathbf{v}}{\sqrt{k}} \right) dt du d\vartheta d\theta \right] d\mathbf{v}, \end{aligned} \quad (71)$$

where now

$$\begin{aligned} \tilde{\Psi}_2 &= t \psi \left(x + \frac{\mathbf{u}}{\sqrt{k}}, x + \left(\theta, \frac{\mathbf{v}}{\sqrt{k}} \right) \right) \\ &+ u \psi \left(\phi_{-(\tau_0+\tau/\sqrt{k})}^X \left(x + \left(\vartheta + \theta, \frac{\mathbf{v}}{\sqrt{k}} \right) \right), x_{\tau_0} + \frac{\mathbf{w}}{\sqrt{k}} \right) - \vartheta, \end{aligned} \quad (72)$$

and the amplitude $\tilde{\mathcal{A}}_2$ is similarly redefined. We may assume without loss that integration in $d\vartheta d\theta$ is compactly supported near the origin.

Lemma 6.1. *Choose $C_0 > 0$. There exist constants $C_1, C_2 > 0$ such that, uniformly in $|\tau| < C_0$, the contribution to the asymptotics of (71) of the locus where $|\theta| > C_1 k^{-7/18}$, $|\vartheta| > C_2 k^{-7/18}$ is $O(k^{-\infty})$.*

Proof. Let $C_1 > 0$ be arbitrary, and suppose $|\theta| > C_1 k^{-7/18}$. Then

$$\text{dist}_X \left(x + \frac{\mathbf{u}}{\sqrt{k}}, x + \left(\theta, \frac{\mathbf{v}}{\sqrt{k}} \right) \right) > \frac{C_1}{2} k^{-7/18}, \quad (73)$$

and therefore

$$|\partial_t \tilde{\Psi}_2| = \left| \psi \left(x + \frac{\mathbf{u}}{\sqrt{k}}, x + \left(\theta, \frac{\mathbf{v}}{\sqrt{k}} \right) \right) \right| > D k^{-7/9}, \quad (74)$$

for some $D > 0$. Integrating by parts in dt , we conclude that uniformly in $|\tau| < C_1 k^{-7/18}$ the contribution of the locus where and $|\theta| > C_1 k^{-7/18}$ to the asymptotics of (71) is $O(k^{-\infty})$.

Now choose $C_2 \gg \max\{C_0, C_1\}$. If $|\tau| < C_0 k^{-7/18}$ and $|\theta| < C_1 k^{-7/18}$, $|\vartheta| > C_2 k^{-7/18}$ then

$$\text{dist}_X \left(\phi_{-(\tau_0 + \tau/\sqrt{k})}^X \left(x + \left(\vartheta + \theta, \frac{\mathbf{v}}{\sqrt{k}} \right) \right), x_{\tau_0} + \frac{\mathbf{w}}{\sqrt{k}} \right) > \frac{C_2}{2} k^{-7/18},$$

and so

$$|\partial_u \tilde{\Psi}_2| = \left| \psi \left(\phi_{-(\tau_0 + \tau/\sqrt{k})}^X \left(x + \left(\vartheta + \theta, \frac{\mathbf{v}}{\sqrt{k}} \right) \right), x_{\tau_0} + \frac{\mathbf{w}}{\sqrt{k}} \right) \right| > D' k^{-7/9}.$$

We now argue as before, using integration by parts in du . □

We may thus introduce in (71) a cut-off of the form $\gamma(k^{7/18} \|(\theta, \vartheta)\|)$, where $\gamma \in \mathcal{C}_0^\infty(\mathbb{R})$ is ≥ 0 and $\equiv 1$ near the origin, perhaps at the cost of losing a rapidly decreasing contribution (in \mathcal{C}^j norm).

With the rescaling $(\theta, \vartheta) \mapsto (\theta, \vartheta)/\sqrt{k}$, we may then rewrite (71) as follows:

$$\begin{aligned} & \tilde{U}_{\tau,k} \left(x + \frac{\mathbf{u}}{\sqrt{k}}, x_{\tau_0} + \frac{\mathbf{w}}{\sqrt{k}} \right) \\ & \sim \frac{k^{1-d}}{2\pi} \int_{\mathbb{C}^d} \left[\int_{1/E}^E \int_{1/E}^E \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} e^{ik\hat{\Psi}_2} \hat{\mathcal{A}}_2 \cdot \nu \left(\frac{\theta}{\sqrt{k}}, \frac{\mathbf{v}}{\sqrt{k}} \right) dt du d\vartheta d\theta \right] d\mathbf{v}, \end{aligned} \quad (75)$$

where $\hat{\Psi}_2$ and $\hat{\mathcal{A}}_2$ are just $\tilde{\Psi}_2$ and $\tilde{\mathcal{A}}_2$ with the previous rescaling inserted, respectively, and in addition a cut-off $\gamma(k^{-1/9} \|(\theta, \vartheta)\|)$ has been incorporated into $\hat{\mathcal{A}}_2$. In particular, integration in $d\theta d\vartheta$ is over a ball centered at the origin in \mathbb{R}^2 , of expanding radius $O(k^{1/9})$.

We have (see §3 of [SZ])

$$\begin{aligned}
& t \psi \left(x + \frac{\mathbf{u}}{\sqrt{k}}, x + \left(\frac{\theta}{\sqrt{k}}, \frac{\mathbf{v}}{\sqrt{k}} \right) \right) \\
&= it \left[1 - e^{-i\theta/\sqrt{k}} \right] - \frac{it}{k} \psi_2(\mathbf{u}, \mathbf{v}) + t R_3 \left(\frac{\mathbf{u}}{\sqrt{k}}, \frac{\mathbf{v}}{\sqrt{k}} \right) e^{-i\theta/\sqrt{k}} \\
&= -\frac{t\theta}{\sqrt{k}} + \frac{it}{2k} \theta^2 - \frac{it}{k} \psi_2(\mathbf{u}, \mathbf{v}) + t R_3 \left(\frac{\theta}{\sqrt{k}}, \frac{\mathbf{u}}{\sqrt{k}}, \frac{\mathbf{v}}{\sqrt{k}} \right).
\end{aligned} \tag{76}$$

Using Corollary 2.2 of [P2], we get from (33):

$$\phi_{-(\tau_0+\tau/\sqrt{k})}^X \left(x + \left(\frac{1}{\sqrt{k}} (\vartheta + \theta), \frac{\mathbf{v}}{\sqrt{k}} \right) \right) \tag{77}$$

$$\begin{aligned}
&= \phi_{-\tau/\sqrt{k}}^X \left(x_{\tau_0} + \left(\frac{1}{\sqrt{k}} (\vartheta + \theta) + R_3 \left(\frac{\mathbf{v}}{\sqrt{k}} \right), \frac{A\mathbf{v}}{\sqrt{k}} + R_2 \left(\frac{\mathbf{v}}{\sqrt{k}} \right) \right) \right) \\
&= x_{\tau_0} + (\Theta_{\tau,k}, \Upsilon_{\tau,k}),
\end{aligned} \tag{78}$$

where

$$\begin{aligned}
\Theta_{\tau,k} &=: \frac{1}{\sqrt{k}} (\tau f(m) + \vartheta + \theta) + \frac{\tau}{k} \omega_m(v_f(m), A\mathbf{v}) + R_3 \left(\frac{\tau}{\sqrt{k}}, \frac{\mathbf{v}}{\sqrt{k}} \right), \\
\Upsilon_{\tau,k} &=: \frac{1}{\sqrt{k}} (A\mathbf{v} - \tau v_f(m)) + R_2 \left(\frac{\tau}{\sqrt{k}}, \frac{\mathbf{v}}{\sqrt{k}} \right).
\end{aligned}$$

Arguing as for (34), we now get

$$\begin{aligned}
& u \psi \left(\phi_{-(\tau_0+\tau/\sqrt{k})}^X \left(x + \left(\frac{1}{\sqrt{k}} (\vartheta + \theta), \frac{\mathbf{v}}{\sqrt{k}} \right) \right), x + \frac{\mathbf{w}}{\sqrt{k}} \right) \\
&= iu \left[1 - e^{i\Theta_{\tau,k}} \right] - \frac{i u}{k} \psi_2(A\mathbf{v} - \tau v_f(m), \mathbf{w}) \\
&\quad + R_3 \left(\frac{\mathbf{v}}{\sqrt{k}}, \frac{\tau}{\sqrt{k}}, \frac{\mathbf{w}}{\sqrt{k}}, \frac{\vartheta}{\sqrt{k}}, \frac{\theta}{\sqrt{k}} \right) \\
&= \frac{u}{\sqrt{k}} (\tau f(m) + \vartheta + \theta) + \frac{u}{k} \left[\tau \omega_m(v_f(m), A\mathbf{v}) + \frac{i}{2} (\tau f(m) + \vartheta + \theta)^2 \right] \\
&\quad - \frac{i u}{k} \psi_2(A\mathbf{v} - \tau v_f(m), \mathbf{w}) + R_3 \left(\frac{\mathbf{v}}{\sqrt{k}}, \frac{\mathbf{w}}{\sqrt{k}}, \frac{\tau}{\sqrt{k}}, \frac{\vartheta}{\sqrt{k}}, \frac{\theta}{\sqrt{k}} \right).
\end{aligned} \tag{79}$$

Inserting (79) into (75), we obtain

$$\begin{aligned}
& \tilde{U}_{\tau,k} \left(x + \frac{\mathbf{u}}{\sqrt{k}}, x_{\tau_0} + \frac{\mathbf{w}}{\sqrt{k}} \right) \\
&\sim \frac{k^{1-d}}{2\pi} \int_{\mathbb{C}^d} \left[\int_{1/E}^E \int_{1/E}^E \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} e^{i\sqrt{k}\Psi_\tau} \mathcal{A}_\tau \cdot \mathcal{V} \left(\frac{\theta}{\sqrt{k}}, \frac{\mathbf{v}}{\sqrt{k}} \right) dt du d\vartheta d\theta \right] d\mathbf{v},
\end{aligned} \tag{80}$$

where

$$\Psi_\tau =: u(\tau f(m) + \vartheta + \theta) - t\theta - \vartheta,$$

while

$$\begin{aligned} \mathcal{A}_\tau = & \exp \left(i\tau \omega_m(v_f(m), A\mathbf{v}) - \frac{t}{2}\theta^2 - \frac{u}{2}(\tau f(m) + \vartheta + \theta)^2 \right) \\ & \cdot \exp \left(\psi_2(\mathbf{u}, \mathbf{v}) + \psi_2(A\mathbf{v} - \tau v_f(m), \mathbf{w}) \right) \cdot \mathcal{A}', \end{aligned} \quad (81)$$

where $\mathcal{A}' =: \mathcal{A} \cdot e^{ikR_3}$.

We may Taylor expand (81) in descending powers of $k^{1/2}$, and regard the inner integral in (80) as an oscillatory integral in \sqrt{k} , with real phase $\Psi_\tau = \Psi_\tau(t, \theta, u, \vartheta)$ depending on the parameter τ . Integrating by parts in $dt du$, one sees that only a bounded neighborhood of the origin in the (θ, ϑ) -plane contributes non-negligibly to the asymptotics (we are assuming $|\tau| < c$ for some $c > 0$); we may then introduce an appropriate cut-off and assume without loss that integration is compactly supported.

Now Ψ_τ has the unique stationary point $P_\tau = (1, 0, 1, -\tau f(m))$, and $\Psi_\tau(P_\tau) = \tau f(m)$. Furthermore, the Hessian there is

$$H(P_\tau) =: \begin{pmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

for every τ , so that its signature is zero. In particular, the stationary phase Lemma yields for the inner integral in (80) an asymptotic expansion in descending powers of $k^{1/2}$. Integration in $d\mathbf{v}$, on the other hand, is over a ball of radius $O(k^{1/9})$, and the expansion may be integrated term by term.

Summing up, we get for (80) an asymptotic expansion of the form

$$\begin{aligned} & \tilde{U}_{\tau,k} \left(x + \frac{\mathbf{u}}{\sqrt{k}}, x_{\tau_0} + \frac{\mathbf{w}}{\sqrt{k}} \right) \\ & \sim e^{i\sqrt{k}\tau f(m)} \left[\varrho_{\tau_0}(m) \left(\frac{k}{\pi} \right)^d \frac{2^d}{\nu(\tau_0, m)} \cdot e^{\mathcal{S}_{\tau_0, m}(\mathbf{u}, \mathbf{w})} + O(k^{d-1/2}) \right], \end{aligned} \quad (82)$$

where the remainder is a function of τ . The expansion may be differentiated in τ , and the leading order term of the derivative at $\tau = 0$ is

$$i\sqrt{k} f(m) \varrho_\tau(m) \left(\frac{k}{\pi} \right)^d \frac{2^d}{\nu(\tau, m)} \cdot e^{\mathcal{S}_{\tau, m}(\mathbf{u}, \mathbf{w})}.$$

The statement follows in view of Theorem 1.2 and (70).

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